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Complete linear descriptions of small asymmetric traveling salesman polytopes

R. Euler[a,*], H. Le Verge[b]

[a] *Laboratoire d'Informatique de Brest (LIBr), Université de Bretagne Occidentale, BP 452, 6 Avenue Le Gorgeu, 29287 Brest Cedex, France*

[b] *IRISA, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France*

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Abstract

Using a refined version of Chernikova's algorithm we determined a complete and irredundant linear description of small asymmetric traveling salesman polytopes. We present such a description for the monotone polytope on 5 nodes consisting of 7615 facet-defining inequalities, and we present 319 015 facet-defining inequalities which, together with 11 equations, fully describe the (non-monotone) asymmetric polytope for 6 nodes.

1. Introduction

Complete linear descriptions of polytopes associated with combinatorial optimization problems are interesting for several reasons, in particular:

- Special classes of facet-defining inequalities can be generalized to be integrated into solution procedures for large instances of the problem.
- In connection with (de-)composition procedures, they can help to design polynomial algorithms for particular (but arbitrary large) instances of the problem. For further details concerning such an approach with respect to the symmetric traveling salesman problem we refer the reader to Cornuéjols et al. [7] and with respect to the maximum independent set problem to Euler and Mahjoub [10]. Such (de-)composition procedures rely heavily on a good knowledge of "irreducible" components, in particular as far as linear descriptions are concerned. It might therefore be interesting to dispose of an efficient computer code which starting out from the solutions of such an irreducible component determines a minimal and complete linear description of the associated polytope.

We believe that the results presented here will contribute to better describe the *general* asymmetric traveling salesman polytope (for previous work see [1, 2, 11–13]) and we

* Corresponding author.

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hope that this will lead to better cutting plane methods for the solution of related applications. It is with these aspects in mind that we conducted our study of small asymmetric traveling salesman polytopes.

The asymmetric traveling salesman polytope on n nodes is defined to be the convex hull of the incidence vectors of all $(n-1)!$ tours in the complete digraph of order n. As in [12], whose notation we adopt throughout this paper, we denote this polytope by $\boldsymbol{P}_T^n$ and recall that its dimension is $n(n-3)+1$. Its monotonization is denoted by $\tilde{\boldsymbol{P}}_T^n$ and consists of the incidence vectors of all graphs which are completable to a tour. The efforts to completely describe $\boldsymbol{P}_T^n$ for small n date back to the years 1953 and 1955 when Heller [14] and Kuhn [15] found such a description for $n=5$ in terms of 9 equations and 390 inequalities (see also [3]).

The case $n=6$ was open for quite some time until Kuhn and Trotter [16] announced a solution in the 1991 *Symposium on Mathematical Programming* in Amsterdam (see also [9]). There have also been efforts in (partially) describing the general polytope $\boldsymbol{P}_T^n$. We refer the interested reader to [11-13, 1, 2].

We obtained our results using an implementation of Chernikova's algorithm [5], which for a given set of extremal rays and vertices determines the associated set of irredundant equations and facet-defining inequalities. The principle of the method is to add one by one the vertices or rays defining the polyhedron and to compute at each step the associated minimal set of facets. It is mainly based on the Fourier–Motzkin elimination method, with an incremental redundancy check as described in [18]: any generated inequality must saturate a minimal number of vertices, and no other valid inequality must saturate a larger set of vertices. In our implementation the facet-vertex incidence matrix is represented in binary in order to take advantage of fast boolean operations. Due to the large amount of memory space needed by such applications and to the slow-down caused by memory page fault, some attention has also been paid to localize the access to this matrix. For further details we refer the reader to [17].

Sections 2 and 3 are devoted to the description of $\tilde{\boldsymbol{P}}_T^5$ and $\boldsymbol{P}_T^6$, respectively. Each class of facet-defining inequalities is represented by its "valued graph" together with some additional information concerning the right-hand side of the inequality, the number of distinct facets in this class, the number of vertices lying on the considered facet, etc. To the best of our ability, we try to cite previously reported facets. Finally, Section 4 contains our conclusion and some comments on related work.

2. The monotone asymmetric traveling salesman polytope on 5 nodes

We first consider the *monotone* asymmetric traveling salesman polytope on 5 nodes, $\tilde{P}_T^5$. By definition, its vertices are all the 525 incidence vectors of arc sets that can be completed to a tour in the complete digraph of order 5. Also observe that $\tilde{\boldsymbol{P}}_T^5$ is full-dimensional and, thus, a minimal linear description of the form $Ax \leqslant b$ contains only inequalities that are unique up to multiplication and whose coefficients are all non-negative, except for the non-negativity inequalities. We will present a complete linear description of $\tilde{P}_T^5$ in terms of 7615 facet-defining inequalities. If we consider two such inequalities to belong to a same class provided one can be obtained by the other by a suitable permutation of the 5 nodes, these inequalities can be divided into 76 classes. Finally, if we merge two classes into one whenever one can be obtained from the other by arc reversal we end up with 51 classes altogether.

We first exhibit for each of the 51 classes its associated "valued graph", i.e. the directed graph on 5 nodes which is associated with one representative inequality and which contains a distinguished arc for every non-zero coefficient of the inequality. Each graph is labeled with "Number/RHS/Facets/Vertices/Tours/Rank/Inv", where:

- "Number" is a class number (see the comments below);
- "RHS" is the right-hand side of the inequalities of one class;
- "Facets" contains the total number of distinct facet-defining inequalities belonging to a class;
- "Vertices" indicates the number of vertices lying on each facet of this class;
- "Tours" denotes the number of tours lying on a facet;
- "Rank" indicates the number of affinely independent tours lying on each facet. If this number is 11, this facet is also a facet of P_T^5;
- "Inv" indicates invariance with respect to arc reversal, by Y standing for invariant, i.e. arc reversal can also be obtained by node permutation, and N standing for non-invariant.

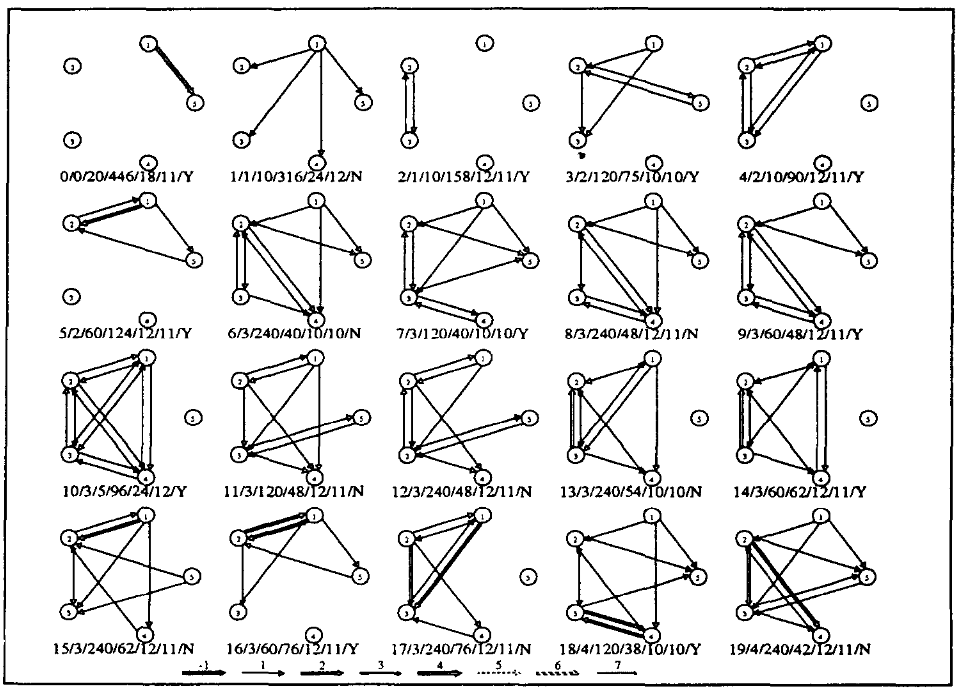

Fig. 1.

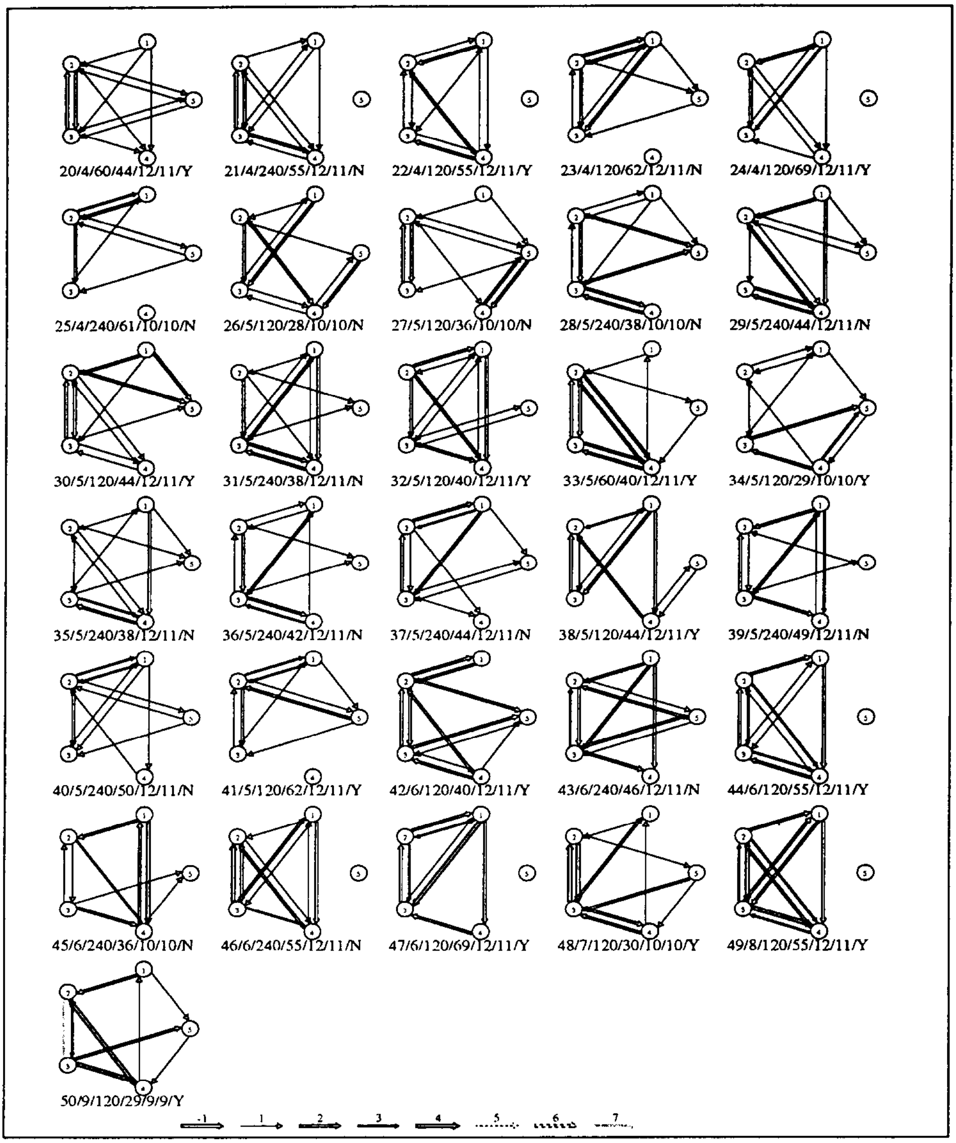

Fig. 2.

The following is a list of previously reported facets:

- Class 0, 1: non-negativity and degree;
- Classes 2, 4, 10: 2, 3 and 4 nodes subtour elimination respectively;
- Class 3: T_2 [12], odd CAT (a) [1];
- Class 5: D_3^+, D_3^- [12, 11];
- Class 9: T_3, generalized odd CAT [12, 1];
- Class 11: odd CAT (b) [1];

- Class 12: C_3^+, C_3^- [12, 11];
- Class 13: lifted subtour;
- Class 14: lifted subtour, (c_3) [12];
- Class 16: E_4, lifted subtour, (c_4) [12];
- Class 17: lifted subtour, D_4^+, D_4^- (c_1) [12].

Summing up the number of different facet-defining inequalities induced by every class we obtain a total number of 7615 linear inequalities fully describing $\tilde{P}_T^5$. Note that, among the above 51 classes, 36 also define facets of P_T^5 and, indeed, can be directly be obtained by the procedure described in [11].

3. The asymmetric traveling salesman polytope on 6 nodes

In this section, we consider P_T^6, which is defined to be the convex hull of the incidence vectors of 120 tours in the complete digraph on 6 nodes. As many other polytopes associated to combinatorial optimization problems the polytope P_T^6 is also symmetric in the sense that the facets containing one specific vertex can be obtained from the facets containing some arbitrary vertex by a suitable permutation of the 6 nodes. So if we denote by x_1 the incidence vector of the tour $\{(1,2),(2,3),(3,4),(4,5),(5,6),(6,1)\}$ and by $x_2, x_3, \ldots, x_{120}$ those associated with the remaining tours, it is sufficient to consider C, the pointed cone induced by the rays $x_2 - x_1, \ldots, x_{120} - x_1$ and the vertex x_1. Note that any ray that is a non-negative linear combination of the remaining ones can be eliminated, and our computational experience showed that it is worthwhile to do so. We used a simplex like procedure to eliminate redundant rays hereby obtaining all those rays $x_i - x_1$, for which x_i is adjacent to x_1 on P_T^6.

We thus reduced the original 119 rays to a set of 110 irredundant rays to which we applied our algorithm. As a result we obtained 11 equations and the remarkable number of 58 799 inequalities fully describing C in an irredundant manner.

The final step consisted in properly identifying all classes of facet-defining inequalities for P_T^6. To this end we projected our description of C into a space of dimension 19 which guarantees the uniqueness (up to multiplication by a scalar) of any facet-defining inequality (see [20] for further details). We then considered two such inequalities to belong to a single class whenever one can be obtained from the other by a permutation of the six nodes. We ended up with 473 classes altogether. To further reduce the number of classes we merged two classes into one whenever one could be obtained from the other by arc reversal.

We describe 287 classes of facet-defining inequalities by exhibiting the associated "valued graph" (i.e. the directed graph on six nodes containing a distinguished arc for every coefficient that is strictly positive) together with the right-hand side of the inequality. Note that each of the representative inequalities has been verified to be "support reduced" (in the sense of [11]) and therefore defines a facet of the monotonization of P_T^6.

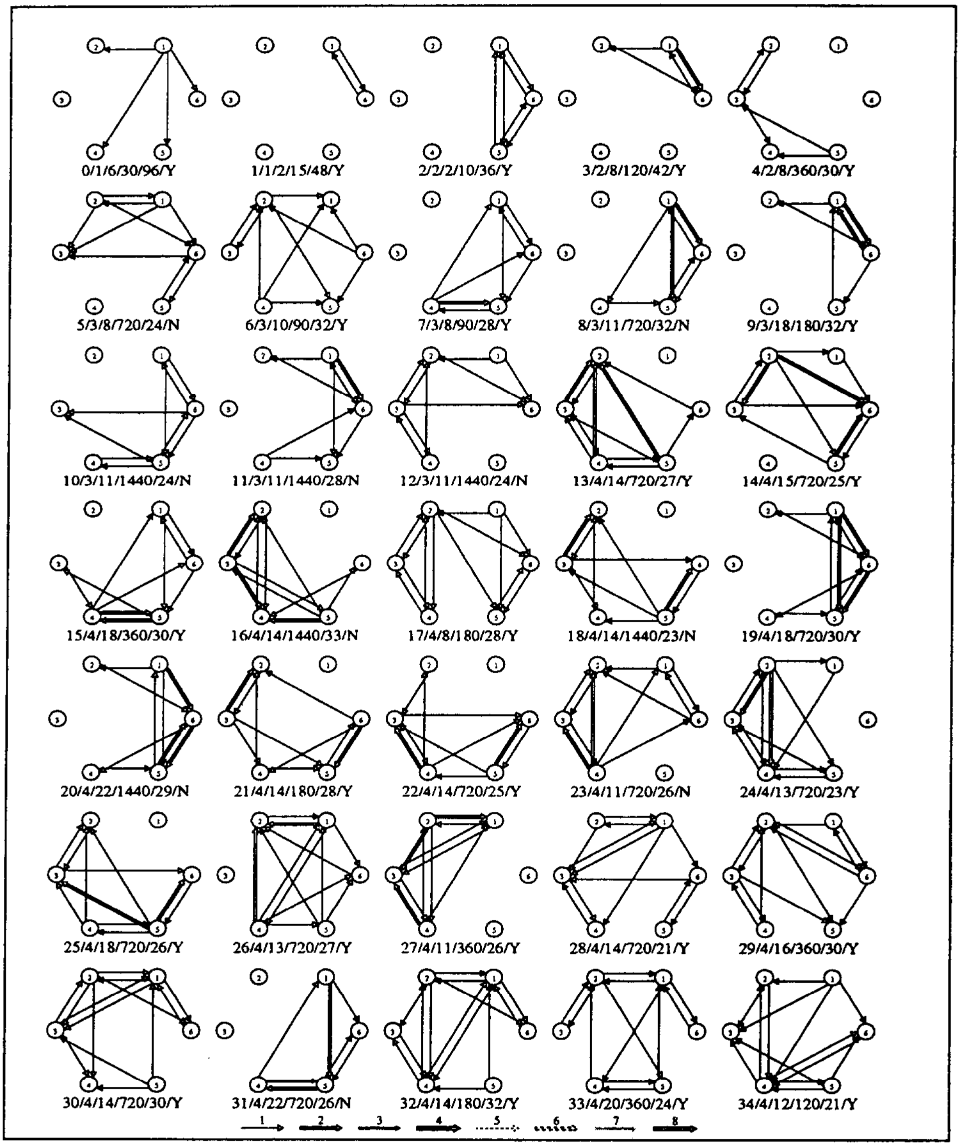
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Fig. 3.

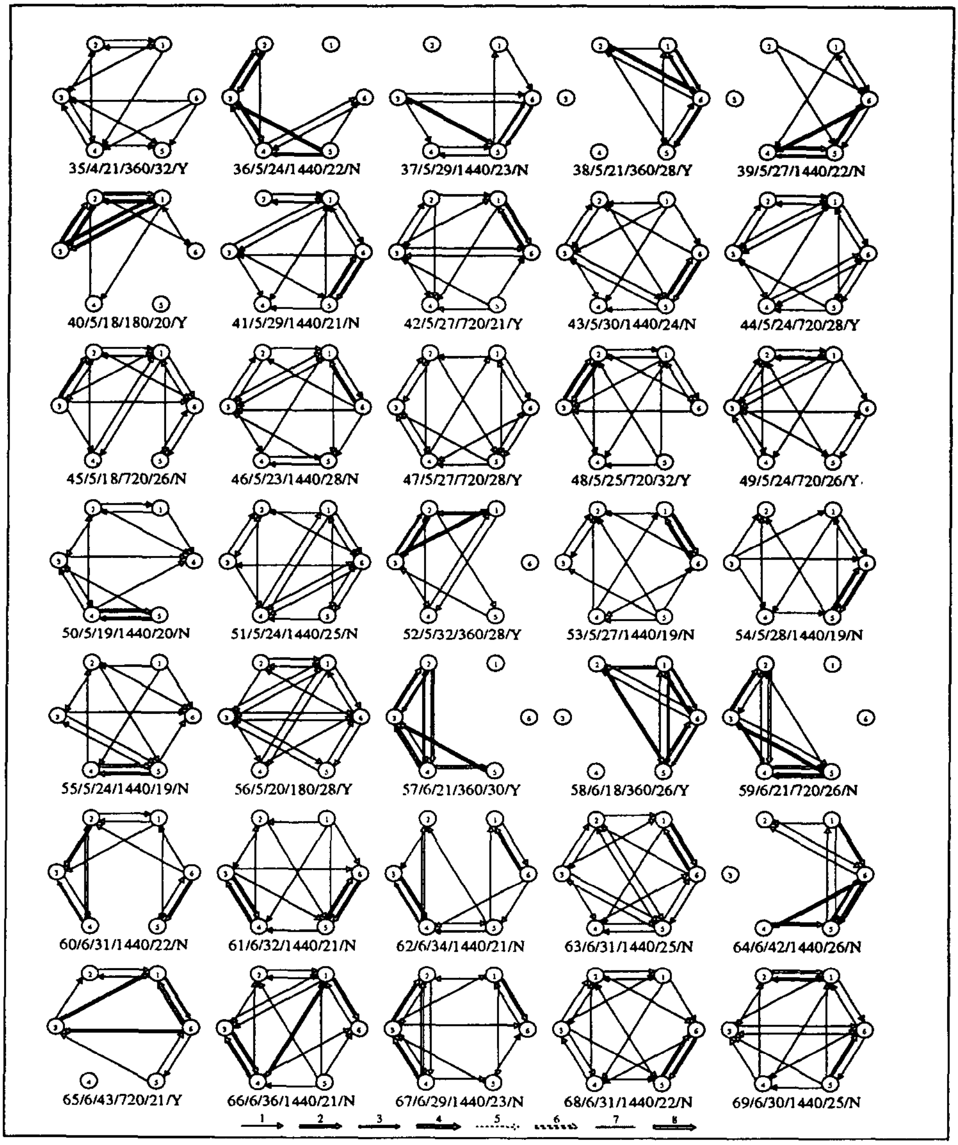
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Fig. 4.

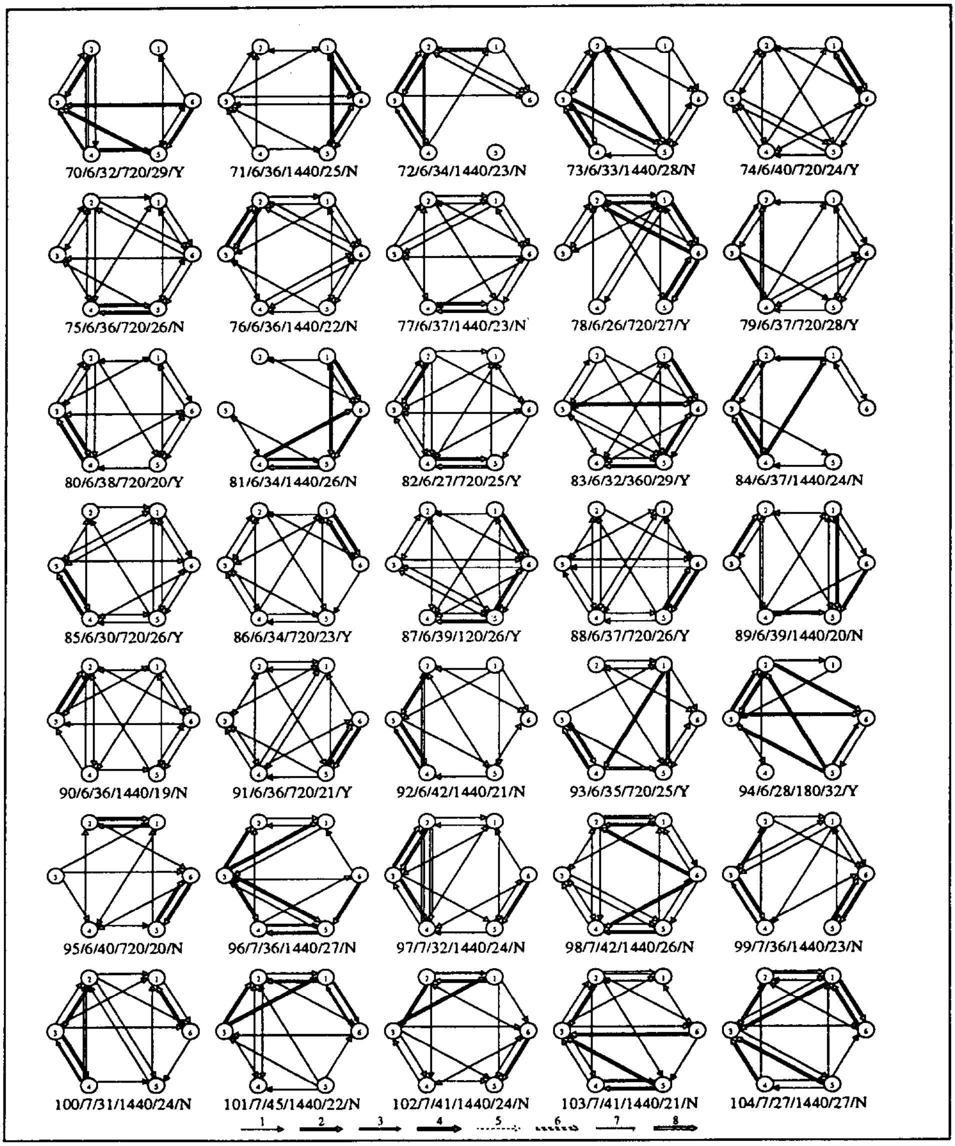
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Fig. 5.

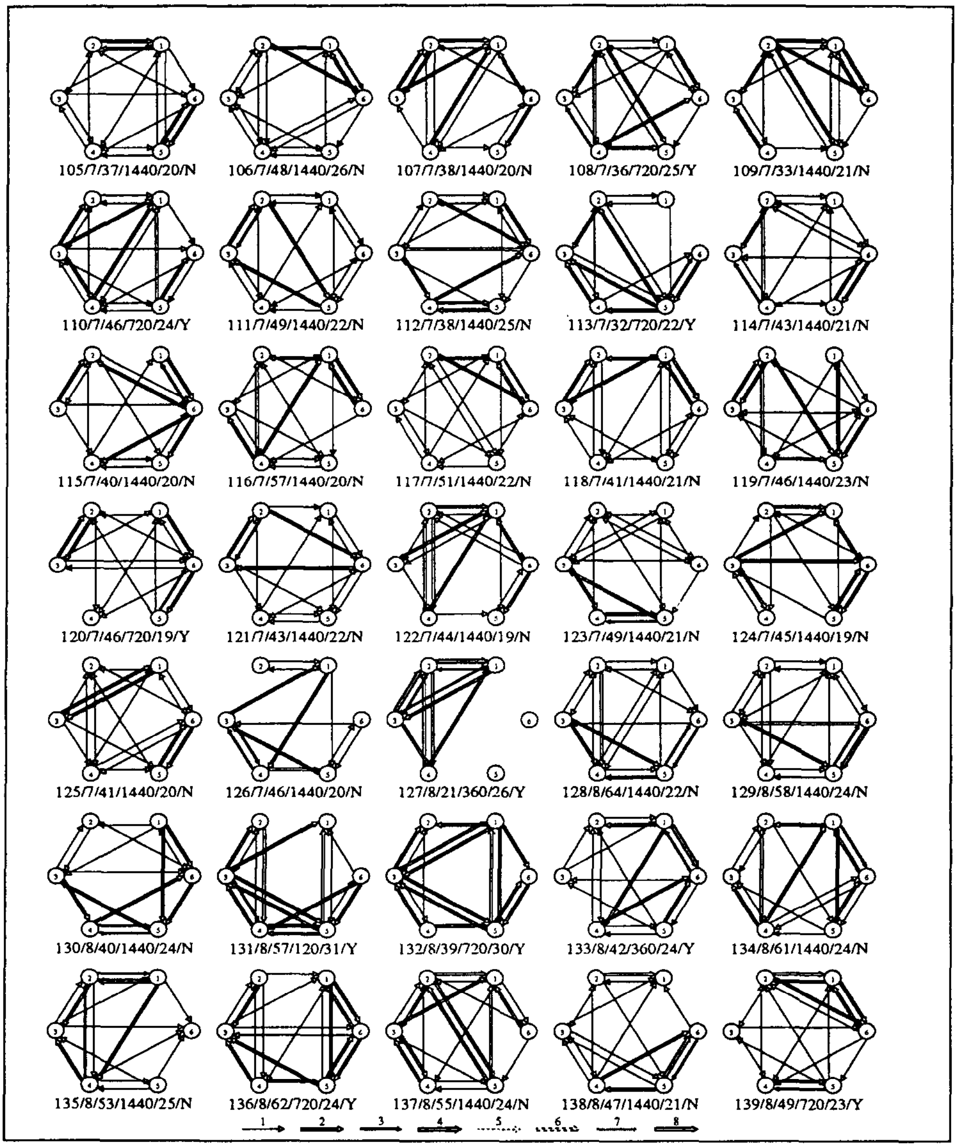
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Fig. 6.

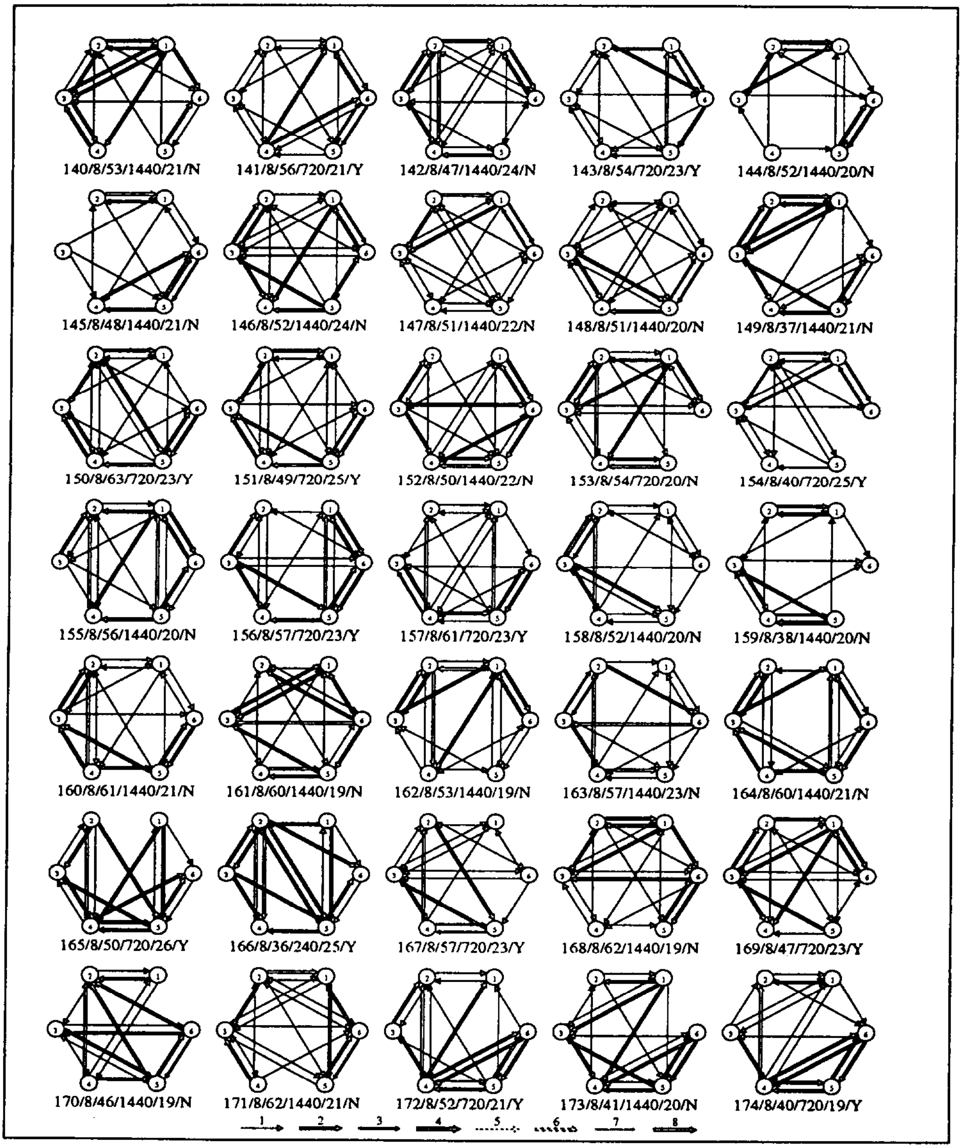
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Fig. 7.

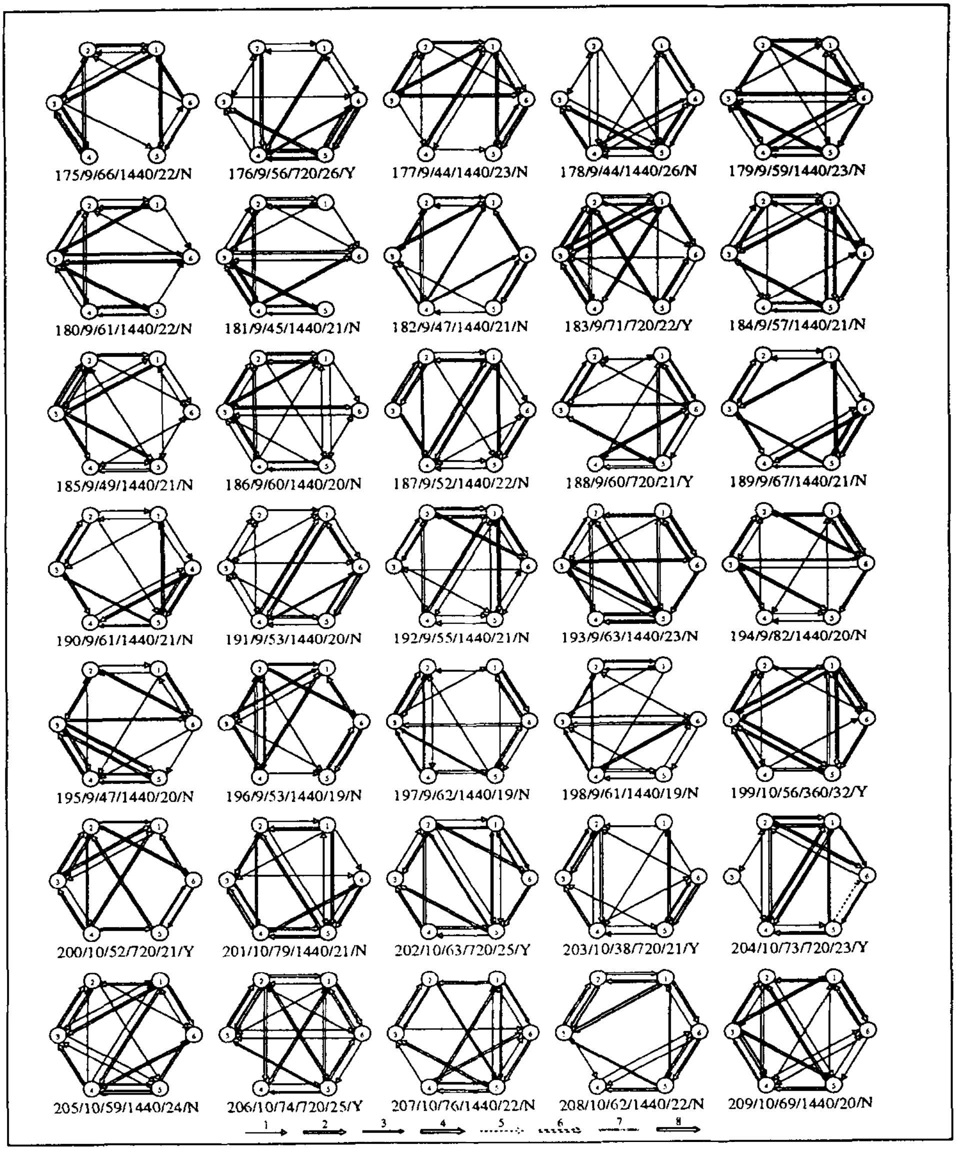
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Fig. 8.

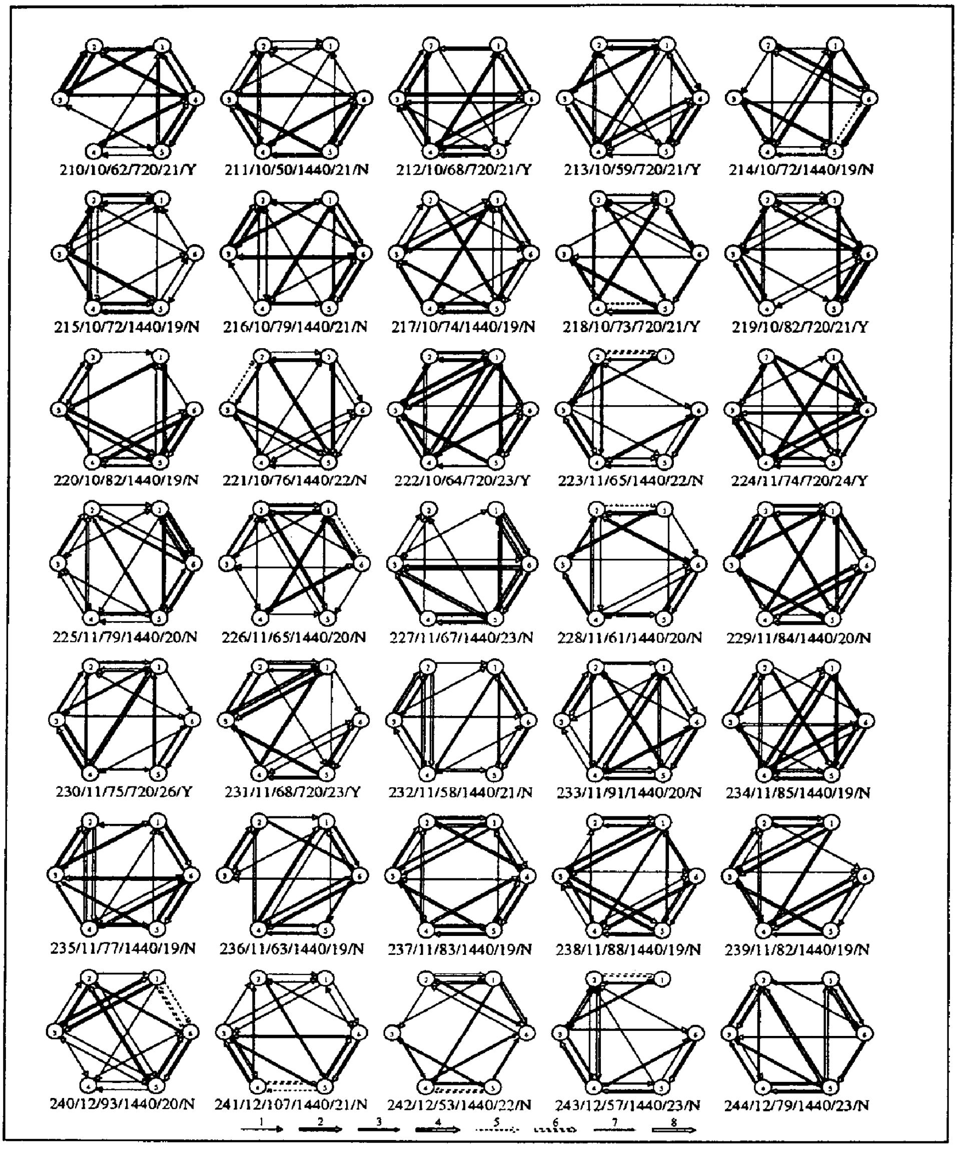
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Fig. 9.

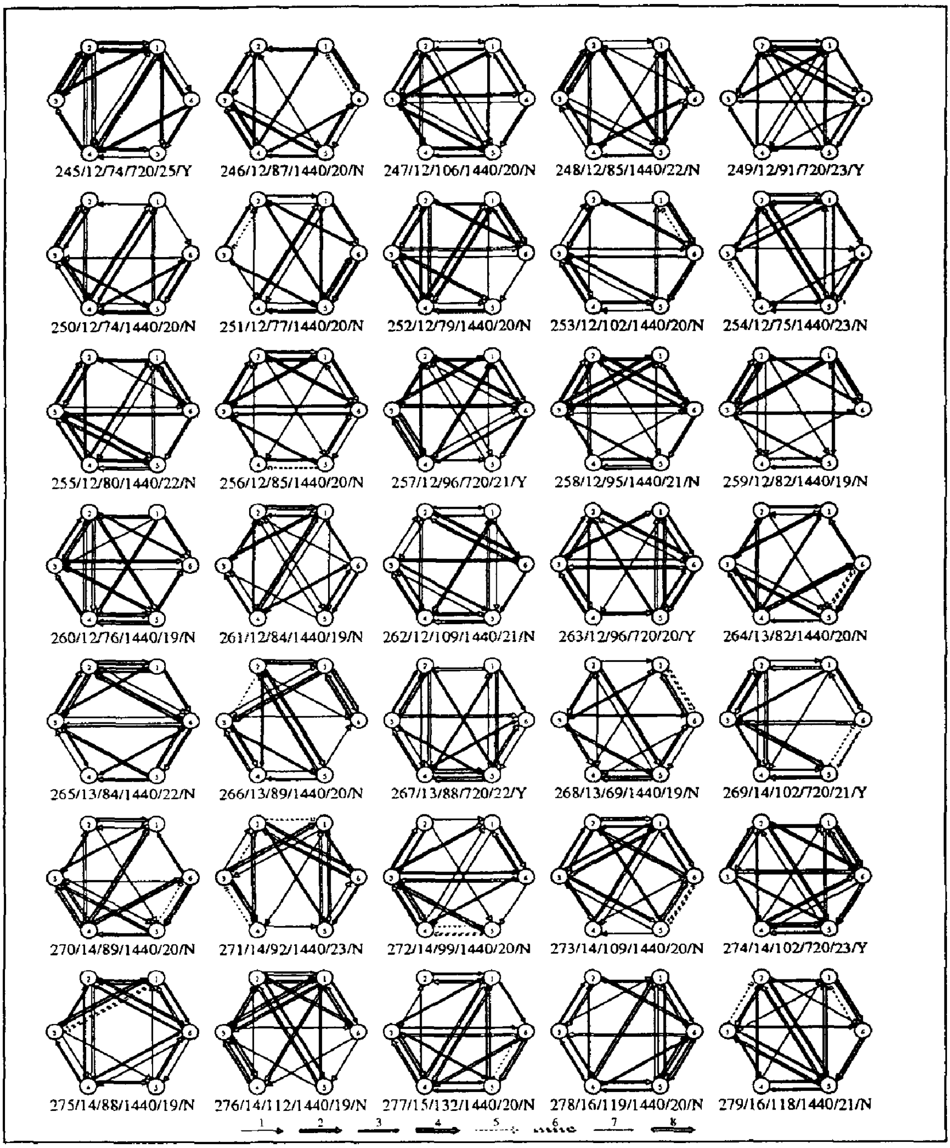
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Fig. 10.

We first exhibit for each of the 287 classes the valued graph associated with one representative facet-defining inequality that has been chosen according to the following criteria:

- the inequality is support reduced;
- its right-hand side is minimum;
- if already known, its form corresponds to that used in the literature;
- inequalities that, after suppression of one isolated node, are equivalent to one defining a facet of $\boldsymbol{P}_T^5$, are represented in the corresponding form.

As for the remaining we have chosen a form in which the highest coefficient is minimum. Each graph is labeled with "Number/RHS/Rep/Facets/Vertices/Inv", where:

- "Number" is a class number (see the comments below);
- "RHS" shows the minimum right-hand side of all the support-reduced inequalities we generated;
- "Rep" indicates the number of distinct support-reduced representations obtained from one facet-defining inequality;
- "Facets" indicates the number of different inequalities forming a class (i.e. those which define the same facet of $\boldsymbol{P}_{\mathrm{T}}^6$ up to permutation of the 6 nodes and up to arc reversal);
- "Vertices" gives the number of vertices lying on each facet of this class;
- "Inv" indicates invariance with respect to arc reversal, by N standing for non invariant and Y for invariant. Note that a class is termed invariant whenever arc reversal can also be obtained by addition of degree inequalities and permutation of nodes.

It turns out that the major part (252 classes among 287 altogether) represents facet-defining inequalities that have been unknown up to 1991 [16]. Here is the list of those classes which correspond to already known facets:

- Class 0: non-negativity;
- Class 1, 2: 2 and 3 nodes subtour elimination;
- Class 3, 13: D_3^+ or D_3^- and D_5^+, D_5^-, (d_1) or (d_2) [12, 11]; note that for n $\geqslant$ 4, any D_{n-1}^- inequality is equivalent to a D_{n-1}^+ inequality.

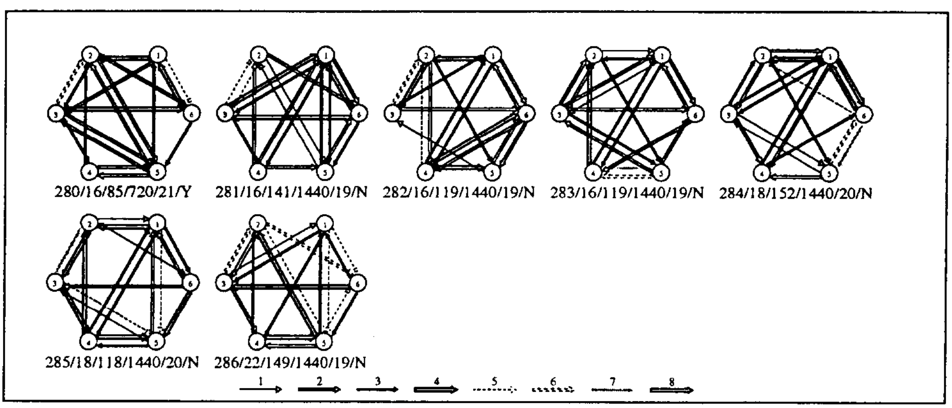

Fig. 11.

Proof. Consider a D_{n-1}^{+} inequality (cf. the notation of [11]) with $i_1 = 1, i_2 = 3, i_3 = 4, \ldots, i_{n-1} = n$ and node 2 as isolated node. Subtracting twice the outdegree equation for node 1, and once the corresponding equation for node n, and adding twice the indegree equation for node 2 and once the corresponding equation for node 3 yields a D_{n-1}^{-} inequality with $i_1 = 2, i_2 = 3, i_3 = 4, \ldots, i_{n-1} = n$ and node 1 as isolated node. □

- Class 10: C_3^+, C_3^- [12, 11];
- Class 4, 5, 6: Odd CAT (a), (b) or (c) and (d) [1];
- Class 7, 8, 9, 14, 15, 16: (c_3), (c_1) or (c_2), (c_4), (d_3) or (d_4), (d_5) and (d_6) [12]. For $n = 6$ and $k = 4$, the lifted subtour inequality (c_3) of [12, p. 209] defines the same facet as the odd CAT inequality (e) of [1, p. 435].
- Class 17: [1, 12];
- Class 18, 19, 20, 21, 22, 23, 24, 25, 26: lifted subtour; observe that for $n = 6$ and $k = 5$, the total number of lifted subtour inequalities defining distinct facets is 16, 12 more than indicated in [12];
- Classes 27, 36, 37, 38, 39, 40, 57, 58, 59, 127: these inequalities are already implicitly known from the description of P_T^5 obtained by [14, 15].

Summing up the number of different facet-defining inequalities induced by every class, we obtain a total number of 319 015 such inequalities describing P_T^6 in a complete and irredundant manner. Also observe that "Rep" indicates for each class the number of different inequalities (we generated) that define the same facet, that are also support reduced and thus give rise to distinct facets of the monotonization of P_T^6. These numbers lead to a lower bound of 17 884 952 different facet-defining inequalities for the monotonization of P_T^6.

4. Conclusions and final remarks

It seems realistic to us that a complete linear description of the monotonization of P_T^6 can be obtained in a similar way. Note however that the number of vertices that are adjacent to a tour, for example, will be much higher and, thus, more computational effort will be necessary.

We conclude with some remarks on related work. By splitting up the cone induced by a given vertex and its adjacent vertices a complete description of the symmetric traveling salesman polytope on 8 nodes has been obtained in [6] hereby detecting 3 new classes of facet-defining inequalities (see also [4] for the case $n = 6$ and $n = 7$). Reinelt [19] applied the same technique to the linear ordering polytope on 7 nodes. He also discovered 3 new classes of facet-defining inequalities. However, in [19] no definite proof of the completeness of the system was given. We adapted our method to the linear ordering polytope and obtained the same results thereby confirming the completeness of Reinelt's description. Finally, we would like to mention the work of Deza et al. [8] on polytopes associated with cut problems on graphs. They have introduced several classes of such polytopes and computed complete and irredundant descriptions for them over graphs of order 4 and 5.

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